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Each day, discovery.



ELTHAM COLLEGE
SIXTH FORM

Additional Work Booklet

Further Mathematics



INSTRUCTIONS

Studying Further Mathematics allows you to explore the Mathematics curriculum to a greater depth and at a faster pace. In addition to being confident in the skills required for A-level Mathematics, to be successful in Further Mathematics you also need to be able to apply a range of techniques within a variety of contexts.

This booklet contains seven sections, which should be completed prior to starting the Sixth Form. During the first week of term there will be a test to assess your understanding and application of this content.

CONTENTS

- Completing the square
- Straight line graphs
- Parallel and perpendicular lines
- Rearranging equations
- Circle theorems
- Trigonometry
- Problem-solving



Completing the square

A LEVEL LINKS

Scheme of work: 1b. Quadratic functions – factorising, solving, graphs and the discriminants

Key points

- Completing the square for a quadratic rearranges $ax^2 + bx + c$ into the form $p(x + q)^2 + r$
- If $a \neq 1$, then factorise using a as a common factor.

Examples

Example 1 Complete the square for the quadratic expression $x^2 + 6x - 2$

$x^2 + 6x - 2$ $= (x + 3)^2 - 9 - 2$ $= (x + 3)^2 - 11$	<p>1 Write $x^2 + bx + c$ in the form $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$</p> <p>2 Simplify</p>
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Example 2 Write $2x^2 - 5x + 1$ in the form $p(x + q)^2 + r$

$2x^2 - 5x + 1$ $= 2\left(x^2 - \frac{5}{2}x\right) + 1$ $= 2\left[\left(x - \frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2\right] + 1$ $= 2\left(x - \frac{5}{4}\right)^2 - \frac{25}{8} + 1$ $= 2\left(x - \frac{5}{4}\right)^2 - \frac{17}{8}$	<p>1 Before completing the square write $ax^2 + bx + c$ in the form $a\left(x^2 + \frac{b}{a}x\right) + c$</p> <p>2 Now complete the square by writing $x^2 - \frac{5}{2}x$ in the form $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$</p> <p>3 Expand the square brackets – don't forget to multiply $\left(\frac{5}{4}\right)^2$ by the factor of 2</p> <p>4 Simplify</p>
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Practice

- Write the following quadratic expressions in the form $(x + p)^2 + q$
 - $x^2 + 4x + 3$
 - $x^2 - 10x - 3$
 - $x^2 - 8x$
 - $x^2 + 6x$
 - $x^2 - 2x + 7$
 - $x^2 + 3x - 2$
- Write the following quadratic expressions in the form $p(x + q)^2 + r$
 - $2x^2 - 8x - 16$
 - $4x^2 - 8x - 16$
 - $3x^2 + 12x - 9$
 - $2x^2 + 6x - 8$
- Complete the square.
 - $2x^2 + 3x + 6$
 - $3x^2 - 2x$
 - $5x^2 + 3x$
 - $3x^2 + 5x + 3$

Extend

- Write $(25x^2 + 30x + 12)$ in the form $(ax + b)^2 + c$.



Answers

1 a $(x+2)^2 - 1$

b $(x-5)^2 - 28$

c $(x-4)^2 - 16$

d $(x+3)^2 - 9$

e $(x-1)^2 + 6$

f $\left(x + \frac{3}{2}\right)^2 - \frac{17}{4}$

2 a $2(x-2)^2 - 24$

b $4(x-1)^2 - 20$

c $3(x+2)^2 - 21$

d $2\left(x + \frac{3}{2}\right)^2 - \frac{25}{2}$

3 a $2\left(x + \frac{3}{4}\right)^2 + \frac{39}{8}$

b $3\left(x - \frac{1}{3}\right)^2 - \frac{1}{3}$

c $5\left(x + \frac{3}{10}\right)^2 - \frac{9}{20}$

d $3\left(x + \frac{5}{6}\right)^2 + \frac{11}{12}$

4 $(5x+3)^2 + 3$



Straight line graphs

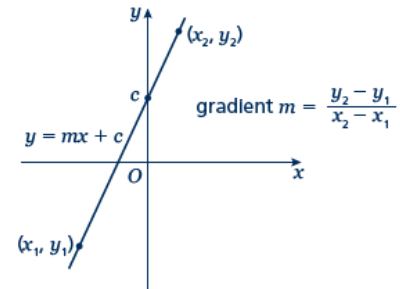
A LEVEL LINKS

Scheme of work: 2a. Straight-line graphs, parallel/perpendicular, length and area problems

Key points

- A straight line has the equation $y = mx + c$, where m is the gradient and c is the y -intercept (where $x = 0$).
- The equation of a straight line can be written in the form $ax + by + c = 0$, where a , b and c are integers.
- When given the coordinates (x_1, y_1) and (x_2, y_2) of two points on a line the gradient is calculated using the

formula $m = \frac{y_2 - y_1}{x_2 - x_1}$



Examples

Example 1 A straight line has gradient $-\frac{1}{2}$ and y -intercept 3.

Write the equation of the line in the form $ax + by + c = 0$.

$$m = -\frac{1}{2} \text{ and } c = 3$$

$$\text{So } y = -\frac{1}{2}x + 3$$

$$\frac{1}{2}x + y - 3 = 0$$

$$x + 2y - 6 = 0$$

- 1 A straight line has equation $y = mx + c$. Substitute the gradient and y -intercept given in the question into this equation.
- 2 Rearrange the equation so all the terms are on one side and 0 is on the other side.
- 3 Multiply both sides by 2 to eliminate the denominator.

Example 2 Find the gradient and the y -intercept of the line with the equation $3y - 2x + 4 = 0$.

$$3y - 2x + 4 = 0$$

$$3y = 2x - 4$$

$$y = \frac{2}{3}x - \frac{4}{3}$$

$$\text{Gradient} = m = \frac{2}{3}$$

$$\text{y-intercept} = c = -\frac{4}{3}$$

- 1 Make y the subject of the equation.
- 2 Divide all the terms by three to get the equation in the form $y = \dots$
- 3 In the form $y = mx + c$, the gradient is m and the y -intercept is c .



Example 3 Find the equation of the line which passes through the point (5, 13) and has gradient 3.

$m = 3$ $y = 3x + c$ $13 = 3 \times 5 + c$ $13 = 15 + c$ $c = -2$ $y = 3x - 2$	<ol style="list-style-type: none">1 Substitute the gradient given in the question into the equation of a straight line $y = mx + c$.2 Substitute the coordinates $x = 5$ and $y = 13$ into the equation.3 Simplify and solve the equation.4 Substitute $c = -2$ into the equation $y = 3x + c$
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Example 4 Find the equation of the line passing through the points with coordinates (2, 4) and (8, 7).

$x_1 = 2, x_2 = 8, y_1 = 4$ and $y_2 = 7$ $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 4}{8 - 2} = \frac{3}{6} = \frac{1}{2}$ $y = \frac{1}{2}x + c$ $4 = \frac{1}{2} \times 2 + c$ $c = 3$ $y = \frac{1}{2}x + 3$	<ol style="list-style-type: none">1 Substitute the coordinates into the equation $m = \frac{y_2 - y_1}{x_2 - x_1}$ to work out the gradient of the line.2 Substitute the gradient into the equation of a straight line $y = mx + c$.3 Substitute the coordinates of either point into the equation.4 Simplify and solve the equation.5 Substitute $c = 3$ into the equation $y = \frac{1}{2}x + c$
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Practice

1 Find the gradient and the y-intercept of the following equations.

a $y = 3x + 5$

b $y = -\frac{1}{2}x - 7$

c $2y = 4x - 3$

d $x + y = 5$

e $2x - 3y - 7 = 0$

f $5x + y - 4 = 0$

Hint

Rearrange the equations to the form $y = mx + c$

2 Copy and complete the table, giving the equation of the line in the form $y = mx + c$.

Gradient	y-intercept	Equation of the line
5	0	
-3	2	
4	-7	



- 3** Find, in the form $ax + by + c = 0$ where a , b and c are integers, an equation for each of the lines with the following gradients and y -intercepts.
- a** gradient $-\frac{1}{2}$, y -intercept -7 **b** gradient 2 , y -intercept 0
- c** gradient $\frac{2}{3}$, y -intercept 4 **d** gradient -1.2 , y -intercept -2
- 4** Write an equation for the line which passes through the point $(2, 5)$ and has gradient 4 .
- 5** Write an equation for the line which passes through the point $(6, 3)$ and has gradient $-\frac{2}{3}$.
- 6** Write an equation for the line passing through each of the following pairs of points.
- a** $(4, 5)$, $(10, 17)$ **b** $(0, 6)$, $(-4, 8)$
- c** $(-1, -7)$, $(5, 23)$ **d** $(3, 10)$, $(4, 7)$

Extend

- 7** The equation of a line is $2y + 3x - 6 = 0$.
Write as much information as possible about this line.



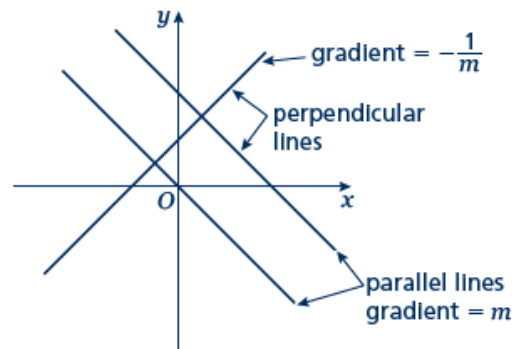
Parallel and perpendicular lines

A LEVEL LINKS

Scheme of work: 2a. Straight-line graphs, parallel/perpendicular, length and area problems

Key points

- When lines are parallel they have the same gradient.
- A line perpendicular to the line with equation $y = mx + c$ has gradient $-\frac{1}{m}$.



Examples

Example 1 Find the equation of the line parallel to $y = 2x + 4$ which passes through the point $(4, 9)$.

$$y = 2x + 4$$

$$m = 2$$

$$y = 2x + c$$

$$9 = 2 \times 4 + c$$

$$9 = 8 + c$$

$$c = 1$$

$$y = 2x + 1$$

- 1 As the lines are parallel they have the same gradient.
- 2 Substitute $m = 2$ into the equation of a straight line $y = mx + c$.
- 3 Substitute the coordinates into the equation $y = 2x + c$
- 4 Simplify and solve the equation.
- 5 Substitute $c = 1$ into the equation $y = 2x + c$

Example 2 Find the equation of the line perpendicular to $y = 2x - 3$ which passes through the point $(-2, 5)$.

$$y = 2x - 3$$

$$m = 2$$

$$-\frac{1}{m} = -\frac{1}{2}$$

$$y = -\frac{1}{2}x + c$$

$$5 = -\frac{1}{2} \times (-2) + c$$

$$5 = 1 + c$$

$$c = 4$$

$$y = -\frac{1}{2}x + 4$$

- 1 As the lines are perpendicular, the gradient of the perpendicular line is $-\frac{1}{m}$.
- 2 Substitute $m = -\frac{1}{2}$ into $y = mx + c$.
- 3 Substitute the coordinates $(-2, 5)$ into the equation $y = -\frac{1}{2}x + c$
- 4 Simplify and solve the equation.
- 5 Substitute $c = 4$ into $y = -\frac{1}{2}x + c$.



Example 3 A line passes through the points (0, 5) and (9, -1).
Find the equation of the line which is perpendicular to the line and passes through its midpoint.

$x_1 = 0, x_2 = 9, y_1 = 5 \text{ and } y_2 = -1$ $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 5}{9 - 0}$ $= \frac{-6}{9} = -\frac{2}{3}$ $-\frac{1}{m} = \frac{3}{2}$ $y = \frac{3}{2}x + c$ $\text{Midpoint} = \left(\frac{0+9}{2}, \frac{5+(-1)}{2} \right) = \left(\frac{9}{2}, 2 \right)$ $2 = \frac{3}{2} \times \frac{9}{2} + c$ $c = -\frac{19}{4}$ $y = \frac{3}{2}x - \frac{19}{4}$	<ol style="list-style-type: none">1 Substitute the coordinates into the equation $m = \frac{y_2 - y_1}{x_2 - x_1}$ to work out the gradient of the line.2 As the lines are perpendicular, the gradient of the perpendicular line is $-\frac{1}{m}$.3 Substitute the gradient into the equation $y = mx + c$.4 Work out the coordinates of the midpoint of the line.5 Substitute the coordinates of the midpoint into the equation.6 Simplify and solve the equation.7 Substitute $c = -\frac{19}{4}$ into the equation $y = \frac{3}{2}x + c$.
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Practice

1 Find the equation of the line parallel to each of the given lines and which passes through each of the given points.

a $y = 3x + 1$ (3, 2)

b $y = 3 - 2x$ (1, 3)

c $2x + 4y + 3 = 0$ (6, -3)

d $2y - 3x + 2 = 0$ (8, 20)

2 Find the equation of the line perpendicular to $y = \frac{1}{2}x - 3$ which passes through the point (-5, 3).

Hint

If $m = \frac{a}{b}$ then the
negative reciprocal

$$-\frac{1}{m} = -\frac{b}{a}$$

3 Find the equation of the line perpendicular to each of the given lines and which passes through each of the given points.

a $y = 2x - 6$ (4, 0)

b $y = -\frac{1}{3}x + \frac{1}{2}$ (2, 13)

c $x - 4y - 4 = 0$ (5, 15)

d $5y + 2x - 5 = 0$ (6, 7)



- 4 In each case find an equation for the line passing through the origin which is also perpendicular to the line joining the two points given.
- a $(4, 3), (-2, -9)$ b $(0, 3), (-10, 8)$

Extend

- 5 Work out whether these pairs of lines are parallel, perpendicular or neither.

a $y = 2x + 3$
 $y = 2x - 7$

b $y = 3x$
 $2x + y - 3 = 0$

c $y = 4x - 3$
 $4y + x = 2$

d $3x - y + 5 = 0$
 $x + 3y = 1$

e $2x + 5y - 1 = 0$
 $y = 2x + 7$

f $2x - y = 6$
 $6x - 3y + 3 = 0$

- 6 The straight line L_1 passes through the points A and B with coordinates $(-4, 4)$ and $(2, 1)$, respectively.

a Find the equation of L_1 in the form $ax + by + c = 0$

The line L_2 is parallel to the line L_1 and passes through the point C with coordinates $(-8, 3)$.

b Find the equation of L_2 in the form $ax + by + c = 0$

The line L_3 is perpendicular to the line L_1 and passes through the origin.

c Find an equation of L_3



Answers

1 a $y = 3x - 7$

b $y = -2x + 5$

c $y = -\frac{1}{2}x$

d $y = \frac{3}{2}x + 8$

2 $y = -2x - 7$

3 a $y = -\frac{1}{2}x + 2$

b $y = 3x + 7$

c $y = -4x + 35$

d $y = \frac{5}{2}x - 8$

4 a $y = -\frac{1}{2}x$

b $y = 2x$

5 a Parallel

b Neither

c Perpendicular

d Perpendicular

e Neither

f Parallel

6 a $x + 2y - 4 = 0$

b $x + 2y + 2 = 0$

c $y = 2x$



Rearranging equations

A LEVEL LINKS

Scheme of work: 6a. Definition, differentiating polynomials, second derivatives

Textbook: Pure Year 1, 12.1 Gradients of curves

Key points

- To change the subject of a formula, get the terms containing the subject on one side and everything else on the other side.
- You may need to factorise the terms containing the new subject.

Examples

Example 1 Make t the subject of the formula $v = u + at$.

$v = u + at$ $v - u = at$ $t = \frac{v - u}{a}$	<ol style="list-style-type: none">1 Get the terms containing t on one side and everything else on the other side.2 Divide throughout by a.
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Example 2 Make t the subject of the formula $r = 2t - \pi t$.

$r = 2t - \pi t$ $r = t(2 - \pi)$ $t = \frac{r}{2 - \pi}$	<ol style="list-style-type: none">1 All the terms containing t are already on one side and everything else is on the other side.2 Factorise as t is a common factor.3 Divide throughout by $2 - \pi$.
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Example 3 Make t the subject of the formula $\frac{t+r}{5} = \frac{3t}{2}$.

$\frac{t+r}{5} = \frac{3t}{2}$ $2t + 2r = 15t$ $2r = 13t$ $t = \frac{2r}{13}$	<ol style="list-style-type: none">1 Remove the fractions first by multiplying throughout by 10.2 Get the terms containing t on one side and everything else on the other side and simplify.3 Divide throughout by 13.
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Example 4 Make t the subject of the formula $r = \frac{3t+5}{t-1}$.

$r = \frac{3t+5}{t-1}$ $r(t-1) = 3t+5$ $rt - r = 3t+5$ $rt - 3t = 5+r$ $t(r-3) = 5+r$ $t = \frac{5+r}{r-3}$	<ol style="list-style-type: none">1 Remove the fraction first by multiplying throughout by $t-1$.2 Expand the brackets.3 Get the terms containing t on one side and everything else on the other side.4 Factorise the LHS as t is a common factor.5 Divide throughout by $r-3$.
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Practice

Change the subject of each formula to the letter given in the brackets.

- 1 $C = \pi d$ [d]
- 2 $P = 2l + 2w$ [w]
- 3 $D = \frac{S}{T}$ [T]
- 4 $p = \frac{q-r}{t}$ [t]
- 5 $u = at - \frac{1}{2}t$ [t]
- 6 $V = ax + 4x$ [x]
- 7 $\frac{y-7x}{2} = \frac{7-2y}{3}$ [y]
- 8 $x = \frac{2a-1}{3-a}$ [a]
- 9 $x = \frac{b-c}{d}$ [d]
- 10 $h = \frac{7g-9}{2+g}$ [g]
- 11 $e(9+x) = 2e+1$ [e]
- 12 $y = \frac{2x+3}{4-x}$ [x]

13 Make r the subject of the following formulae.

a $A = \pi r^2$ b $V = \frac{4}{3}\pi r^3$ c $P = \pi r + 2r$ d $V = \frac{2}{3}\pi r^2 h$

14 Make x the subject of the following formulae.

a $\frac{xy}{z} = \frac{ab}{cd}$ b $\frac{4\pi cx}{d} = \frac{3z}{py^2}$

15 Make $\sin B$ the subject of the formula $\frac{a}{\sin A} = \frac{b}{\sin B}$

16 Make $\cos B$ the subject of the formula $b^2 = a^2 + c^2 - 2ac \cos B$.

Extend

17 Make x the subject of the following equations.

a $\frac{p}{q}(sx+t) = x-1$ b $\frac{p}{q}(ax+2y) = \frac{3p}{q^2}(x-y)$



Answers

$$1 \quad d = \frac{C}{\pi}$$

$$2 \quad w = \frac{P-2l}{2}$$

$$3 \quad T = \frac{S}{D}$$

$$4 \quad t = \frac{q-r}{p}$$

$$5 \quad t = \frac{2u}{2a-1}$$

$$6 \quad x = \frac{V}{a+4}$$

$$7 \quad y = 2 + 3x$$

$$8 \quad a = \frac{3x+1}{x+2}$$

$$9 \quad d = \frac{b-c}{x}$$

$$10 \quad g = \frac{2h+9}{7-h}$$

$$11 \quad e = \frac{1}{x+7}$$

$$12 \quad x = \frac{4y-3}{2+y}$$

$$13 \quad \text{a} \quad r = \sqrt{\frac{A}{\pi}}$$

$$\text{b} \quad r = \sqrt[3]{\frac{3V}{4\pi}}$$

$$\text{c} \quad r = \frac{P}{\pi+2}$$

$$\text{d} \quad r = \sqrt{\frac{3V}{2\pi h}}$$

$$14 \quad \text{a} \quad x = \frac{abz}{cdy}$$

$$\text{b} \quad x = \frac{3dz}{4\pi cpy^2}$$

$$15 \quad \sin B = \frac{b \sin A}{a}$$

$$16 \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$17 \quad \text{a} \quad x = \frac{q+pt}{q-ps}$$

$$\text{b} \quad x = \frac{3py+2pqy}{3p-apq} = \frac{y(3+2q)}{3-aq}$$



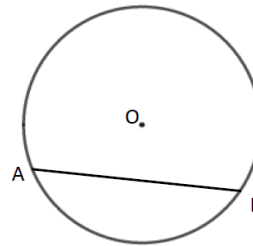
Circle theorems

A LEVEL LINKS

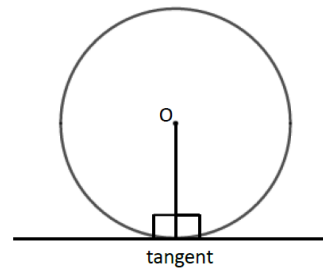
Scheme of work: 2b. Circles – equation of a circle, geometric problems on a grid

Key points

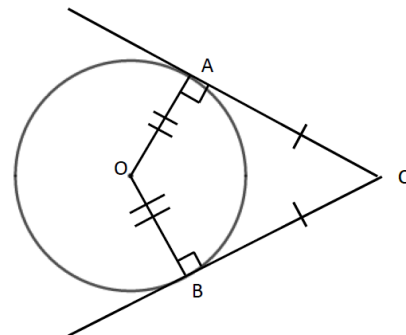
- A chord is a straight line joining two points on the circumference of a circle.
So AB is a chord.



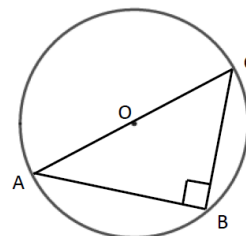
- A tangent is a straight line that touches the circumference of a circle at only one point.
The angle between a tangent and the radius is 90° .



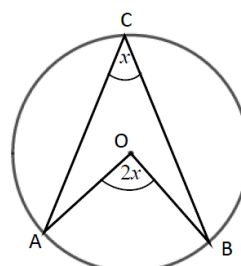
- Two tangents on a circle that meet at a point outside the circle are equal in length.
So $AC = BC$.



- The angle in a semicircle is a right angle.
So angle $ABC = 90^\circ$.

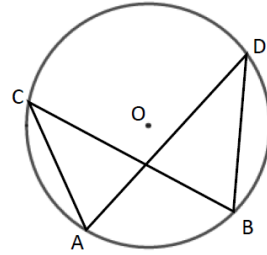


- When two angles are subtended by the same arc, the angle at the centre of a circle is twice the angle at the circumference.
So angle $AOB = 2 \times$ angle ACB .

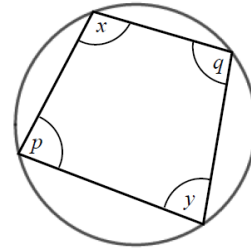




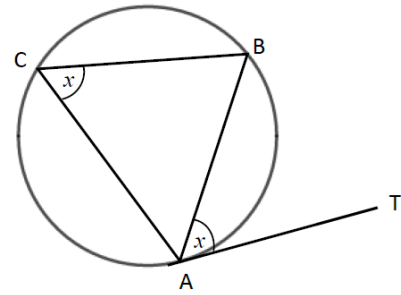
- Angles subtended by the same arc at the circumference are equal. This means that angles in the same segment are equal.
So angle $ACB = \text{angle } ADB$ and
angle $CAD = \text{angle } CBD$.



- A cyclic quadrilateral is a quadrilateral with all four vertices on the circumference of a circle.
Opposite angles in a cyclic quadrilateral total 180° .
So $x + y = 180^\circ$ and $p + q = 180^\circ$.

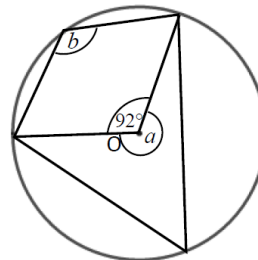


- The angle between a tangent and chord is equal to the angle in the alternate segment, this is known as the alternate segment theorem.
So angle $BAT = \text{angle } ACB$.



Examples

- Example 1** Work out the size of each angle marked with a letter.
Give reasons for your answers.



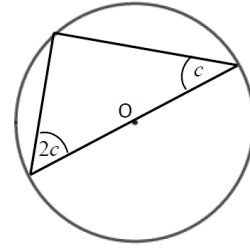
Angle $a = 360^\circ - 92^\circ$
 $= 268^\circ$
as the angles in a full turn total 360° .

Angle $b = 268^\circ \div 2$
 $= 134^\circ$
as when two angles are subtended by the same arc, the angle at the centre of a circle is twice the angle at the circumference.

- 1 The angles in a full turn total 360° .
- 2 Angles a and b are subtended by the same arc, so angle b is half of angle a .



Example 2 Work out the size of the angles in the triangle.
Give reasons for your answers.



Angles are 90° , $2c$ and c .

$$90^\circ + 2c + c = 180^\circ$$

$$90^\circ + 3c = 180^\circ$$

$$3c = 90^\circ$$

$$c = 30^\circ$$

$$2c = 60^\circ$$

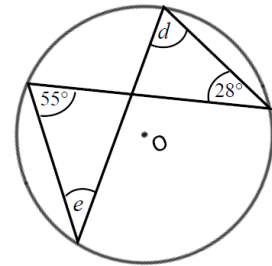
The angles are 30° , 60° and 90° as the angle in a semi-circle is a right angle and the angles in a triangle total 180° .

1 The angle in a semicircle is a right angle.

2 Angles in a triangle total 180° .

3 Simplify and solve the equation.

Example 3 Work out the size of each angle marked with a letter.
Give reasons for your answers.



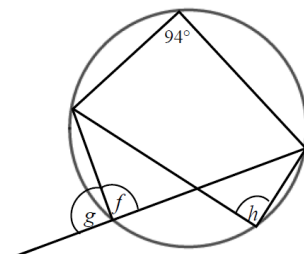
Angle $d = 55^\circ$ as angles subtended by the same arc are equal.

Angle $e = 28^\circ$ as angles subtended by the same arc are equal.

1 Angles subtended by the same arc are equal so angle 55° and angle d are equal.

2 Angles subtended by the same arc are equal so angle 28° and angle e are equal.

Example 4 Work out the size of each angle marked with a letter.
Give reasons for your answers.



Angle $f = 180^\circ - 94^\circ$
 $= 86^\circ$
as opposite angles in a cyclic quadrilateral total 180° .

1 Opposite angles in a cyclic quadrilateral total 180° so angle 94° and angle f total 180° .

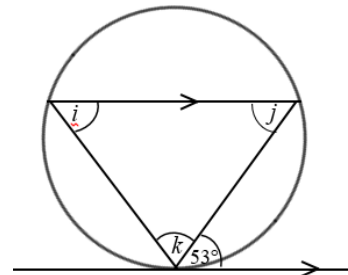
(continued on next page)



<p>Angle $g = 180^\circ - 86^\circ$ $= 84^\circ$ as angles on a straight line total 180°.</p> <p>Angle $h = \text{angle } f = 86^\circ$ as angles subtended by the same arc are equal.</p>	<p>2 Angles on a straight line total 180° so angle f and angle g total 180°.</p> <p>3 Angles subtended by the same arc are equal so angle f and angle h are equal.</p>
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Example 5

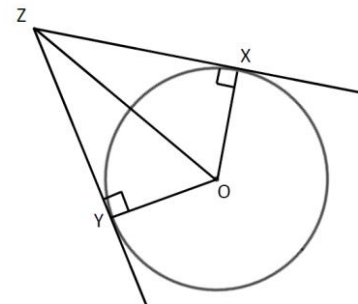
Work out the size of each angle marked with a letter.
Give reasons for your answers.



<p>Angle $i = 53^\circ$ because of the alternate segment theorem.</p> <p>Angle $j = 53^\circ$ because it is the alternate angle to 53°.</p> <p>Angle $k = 180^\circ - 53^\circ - 53^\circ$ $= 74^\circ$ as angles in a triangle total 180°.</p>	<p>1 The angle between a tangent and chord is equal to the angle in the alternate segment.</p> <p>2 As there are two parallel lines, angle 53° is equal to angle j because they are alternate angles.</p> <p>3 The angles in a triangle total 180°, so $i + j + k = 180^\circ$.</p>
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Example 6

XZ and YZ are two tangents to a circle with centre O.
Prove that triangles XZO and YZO are congruent.



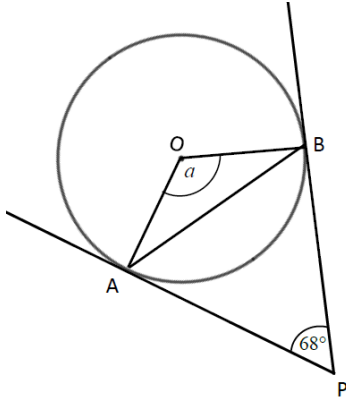
<p>Angle $OXZ = 90^\circ$ and angle $OYZ = 90^\circ$ as the angles in a semicircle are right angles.</p> <p>OZ is a common line and is the hypotenuse in both triangles.</p> <p>$OX = OY$ as they are radii of the same circle.</p> <p>So triangles XZO and YZO are congruent, RHS.</p>	<p>For two triangles to be congruent you need to show one of the following.</p> <ul style="list-style-type: none"> • All three corresponding sides are equal (SSS). • Two corresponding sides and the included angle are equal (SAS). • One side and two corresponding angles are equal (ASA). • A right angle, hypotenuse and a shorter side are equal (RHS).
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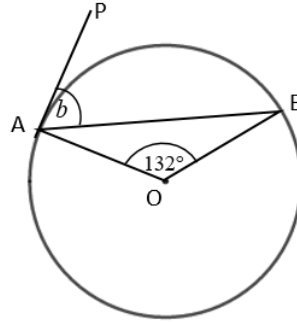
Practice

- 1 Work out the size of each angle marked with a letter.
Give reasons for your answers.

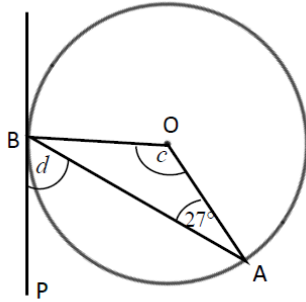
a



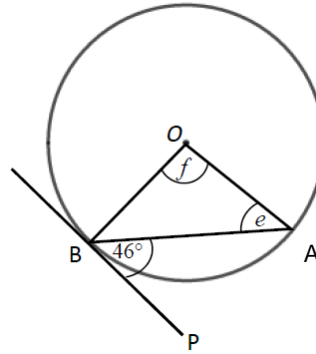
b



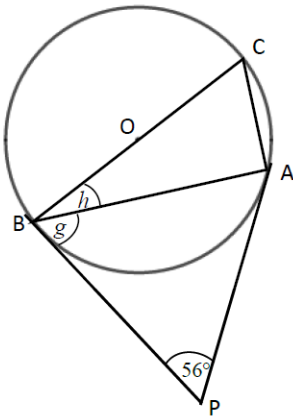
c



d

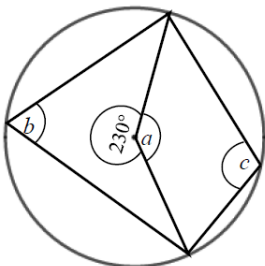


e

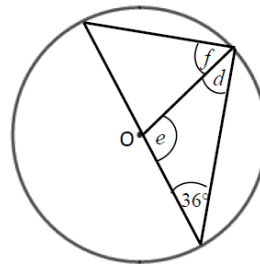


- 2 Work out the size of each angle marked with a letter.
Give reasons for your answers.

a

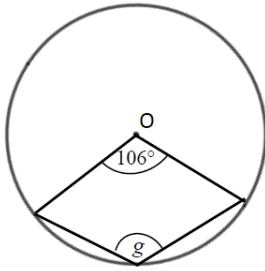


b





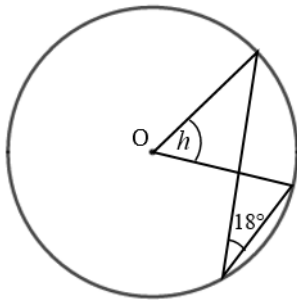
c



Hint

The reflex angle at point O and angle g are subtended by the same arc. So the reflex angle is twice the size of angle g .

d

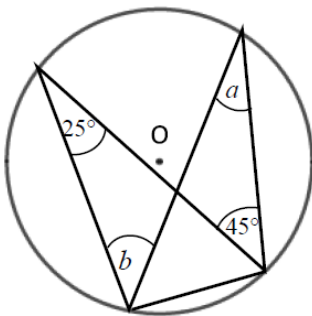


Hint

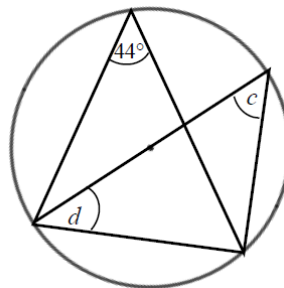
Angle 18° and angle h are subtended by the same arc.

3 Work out the size of each angle marked with a letter.
Give reasons for your answers.

a



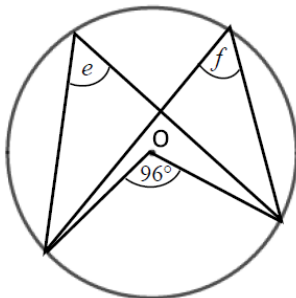
b



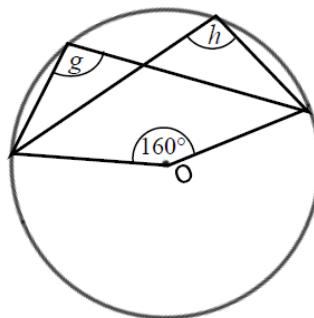
Hint

One of the angles is in a semicircle.

c



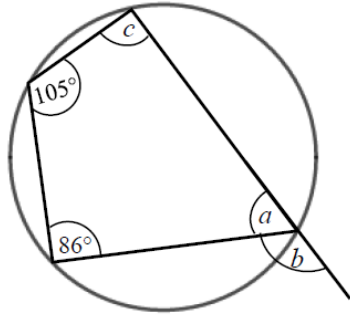
d





- 4 Work out the size of each angle marked with a letter.
Give reasons for your answers.

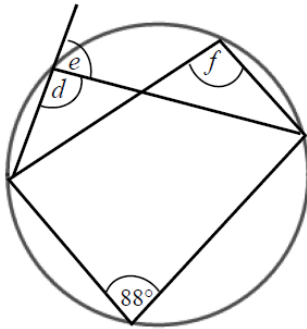
a



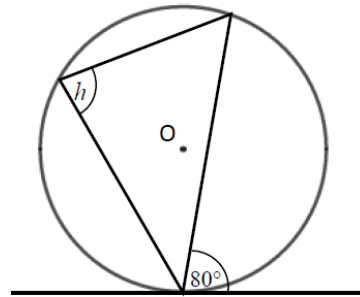
Hint

An exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

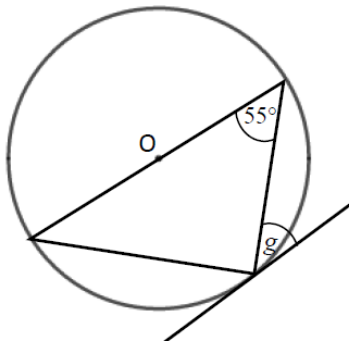
b



c



d



Hint

One of the angles is in a semicircle.

Extend

- 5 Prove the alternate segment theorem.



Answers

- 1**
- a** $a = 112^\circ$, angle OAP = angle OBP = 90° and angles in a quadrilateral total 360° .
- b** $b = 66^\circ$, triangle OAB is isosceles, Angle OAP = 90° as AP is tangent to the circle.
- c** $c = 126^\circ$, triangle OAB is isosceles.
 $d = 63^\circ$, Angle OBP = 90° as BP is tangent to the circle.
- d** $e = 44^\circ$, the triangle is isosceles, so angles e and angle OBA are equal. The angle OBP = 90° as BP is tangent to the circle.
 $f = 92^\circ$, the triangle is isosceles.
- e** $g = 62^\circ$, triangle ABP is isosceles as AP and BP are both tangents to the circle.
 $h = 28^\circ$, the angle OBP = 90° .
- 2**
- a** $a = 130^\circ$, angles in a full turn total 360° .
 $b = 65^\circ$, the angle at the centre of a circle is twice the angle at the circumference.
 $c = 115^\circ$, opposite angles in a cyclic quadrilateral total 180° .
- b** $d = 36^\circ$, isosceles triangle.
 $e = 108^\circ$, angles in a triangle total 180° .
 $f = 54^\circ$, angle in a semicircle is 90° .
- c** $g = 127^\circ$, angles at a full turn total 360° , the angle at the centre of a circle is twice the angle at the circumference.
- d** $h = 36^\circ$, the angle at the centre of a circle is twice the angle at the circumference.
- 3**
- a** $a = 25^\circ$, angles in the same segment are equal.
 $b = 45^\circ$, angles in the same segment are equal.
- b** $c = 44^\circ$, angles in the same segment are equal.
 $d = 46^\circ$, the angle in a semicircle is 90° and the angles in a triangle total 180° .
- c** $e = 48^\circ$, the angle at the centre of a circle is twice the angle at the circumference.
 $f = 48^\circ$, angles in the same segment are equal.
- d** $g = 100^\circ$, angles at a full turn total 360° , the angle at the centre of a circle is twice the angle at the circumference.
 $h = 100^\circ$, angles in the same segment are equal.
- 4**
- a** $a = 75^\circ$, opposite angles in a cyclic quadrilateral total 180° .
 $b = 105^\circ$, angles on a straight line total 180° .
 $c = 94^\circ$, opposite angles in a cyclic quadrilateral total 180° .
- b** $d = 92^\circ$, opposite angles in a cyclic quadrilateral total 180° .
 $e = 88^\circ$, angles on a straight line total 180° .
 $f = 92^\circ$, angles in the same segment are equal.
- c** $h = 80^\circ$, alternate segment theorem.
- d** $g = 35^\circ$, alternate segment theorem and the angle in a semicircle is 90° .



5 Angle $BAT = x$.

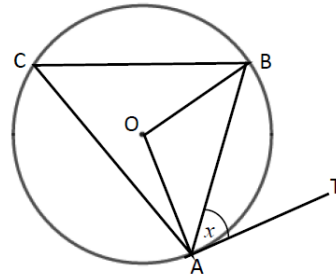
Angle $OAB = 90^\circ - x$ because the angle between the tangent and the radius is 90° .

$OA = OB$ because radii are equal.

Angle $OAB =$ angle OBA because the base of isosceles triangles are equal.

Angle $AOB = 180^\circ - (90^\circ - x) - (90^\circ - x) = 2x$ because angles in a triangle total 180° .

Angle $ACB = 2x \div 2 = x$ because the angle at the centre is twice the angle at the circumference.





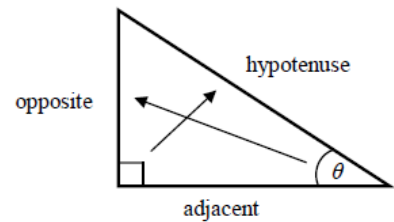
Trigonometry

A LEVEL LINKS

Scheme of work: 4a. Trigonometric ratios and graphs

Key points

- In a right-angled triangle:
 - the side opposite the right angle is called the hypotenuse
 - the side opposite the angle ϑ is called the opposite
 - the side next to the angle ϑ is called the adjacent.



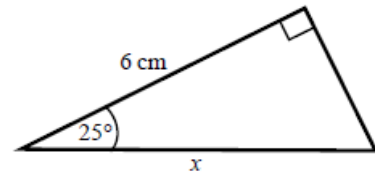
- In a right-angled triangle:
 - the ratio of the opposite side to the hypotenuse is the sine of angle ϑ , $\sin \theta = \frac{\text{opp}}{\text{hyp}}$
 - the ratio of the adjacent side to the hypotenuse is the cosine of angle ϑ , $\cos \theta = \frac{\text{adj}}{\text{hyp}}$
 - the ratio of the opposite side to the adjacent side is the tangent of angle ϑ , $\tan \theta = \frac{\text{opp}}{\text{adj}}$
- If the lengths of two sides of a right-angled triangle are given, you can find a missing angle using the inverse trigonometric functions: \sin^{-1} , \cos^{-1} , \tan^{-1} .
- The sine, cosine and tangent of some angles may be written exactly.

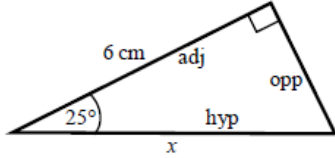
	0	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	



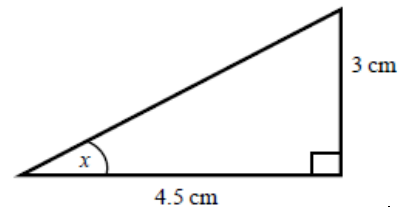
Examples

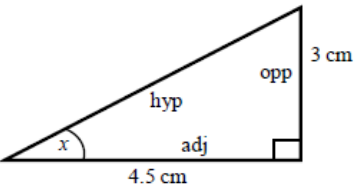
Example 1 Calculate the length of side x .
Give your answer correct to 3 significant figures.



 $\cos \theta = \frac{\text{adj}}{\text{hyp}}$ $\cos 25^\circ = \frac{6}{x}$ $x = \frac{6}{\cos 25^\circ}$ $x = 6.620\ 267\ 5\dots$ $x = 6.62\ \text{cm}$	<ol style="list-style-type: none">1 Always start by labelling the sides.2 You are given the adjacent and the hypotenuse so use the cosine ratio.3 Substitute the sides and angle into the cosine ratio.4 Rearrange to make x the subject.5 Use your calculator to work out $6 \div \cos 25^\circ$.6 Round your answer to 3 significant figures and write the units in your answer.
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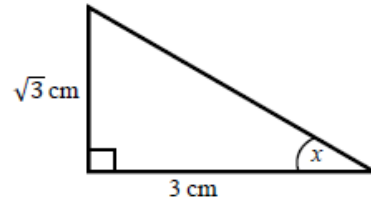
Example 2 Calculate the size of angle x .
Give your answer correct to 3 significant figures.



 $\tan \theta = \frac{\text{opp}}{\text{adj}}$ $\tan x = \frac{3}{4.5}$ $x = \tan^{-1}\left(\frac{3}{4.5}\right)$ $x = 33.690\ 067\ 5\dots$ $x = 33.7^\circ$	<ol style="list-style-type: none">1 Always start by labelling the sides.2 You are given the opposite and the adjacent so use the tangent ratio.3 Substitute the sides and angle into the tangent ratio.4 Use \tan^{-1} to find the angle.5 Use your calculator to work out $\tan^{-1}(3 \div 4.5)$.6 Round your answer to 3 significant figures and write the units in your answer.
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Example 3 Calculate the exact size of angle x .

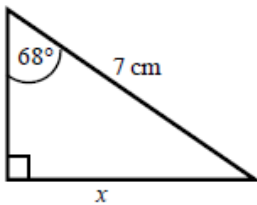


$\tan \theta = \frac{\text{opp}}{\text{adj}}$ $\tan x = \frac{\sqrt{3}}{3}$ $x = 30^\circ$	<ol style="list-style-type: none">1 Always start by labelling the sides.2 You are given the opposite and the adjacent so use the tangent ratio.3 Substitute the sides and angle into the tangent ratio.4 Use the table from the key points to find the angle.
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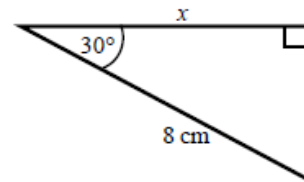
Practice

1 Calculate the length of the unknown side in each triangle. Give your answers correct to 3 significant figures.

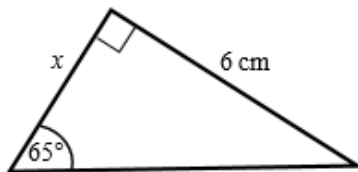
a



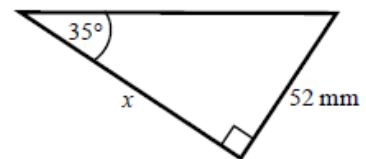
b



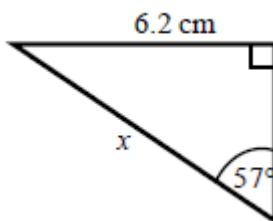
c



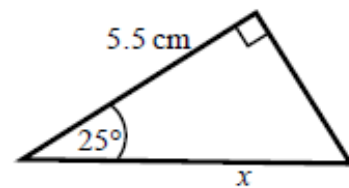
d



e



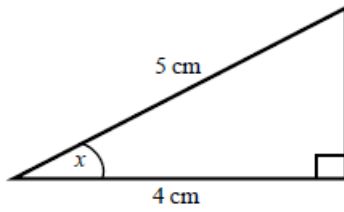
f



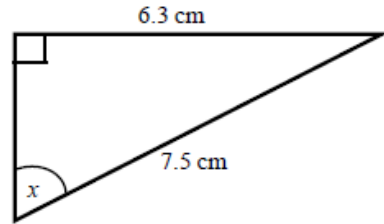


- 2 Calculate the size of angle x in each triangle. Give your answers correct to 1 decimal place.

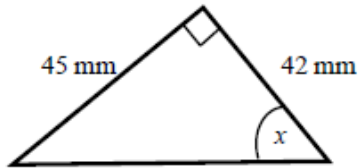
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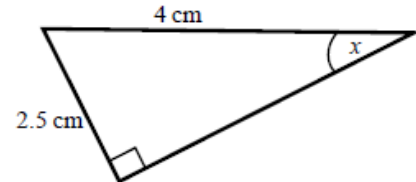
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c



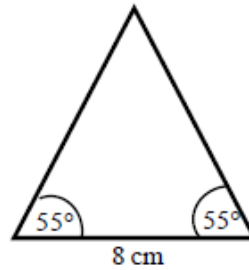
d



- 3 Work out the height of the isosceles triangle. Give your answer correct to 3 significant figures.

Hint:

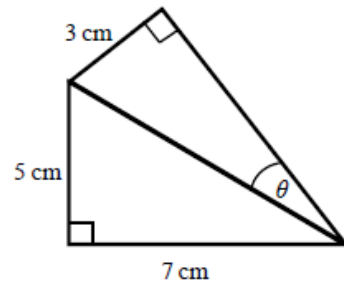
Split the triangle into two right-angled triangles.



- 4 Calculate the size of angle θ . Give your answer correct to 1 decimal place.

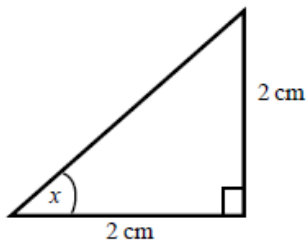
Hint:

First work out the length of the common side to both triangles, leaving your answer in surd form.

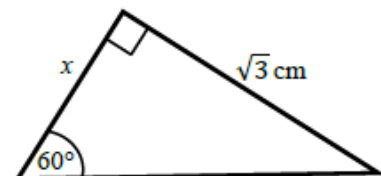


- 5 Find the exact value of x in each triangle.

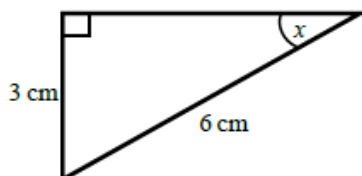
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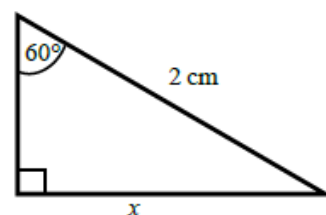
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The cosine rule

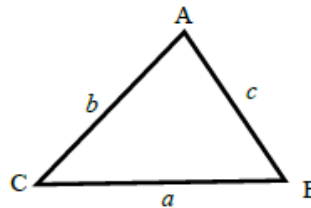
A LEVEL LINKS

Scheme of work: 4a. Trigonometric ratios and graphs

Textbook: Pure Year 1, 9.1 The cosine rule

Key points

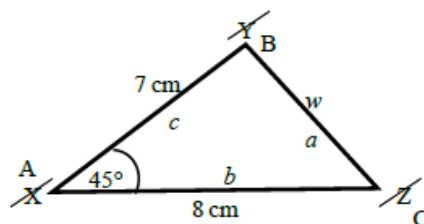
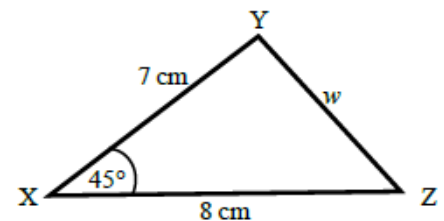
- a is the side opposite angle A .
- b is the side opposite angle B .
- c is the side opposite angle C .



- You can use the cosine rule to find the length of a side when two sides and the included angle are given.
- To calculate an unknown side use the formula $a^2 = b^2 + c^2 - 2bc \cos A$.
- Alternatively, you can use the cosine rule to find an unknown angle if the lengths of all three sides are given.
- To calculate an unknown angle use the formula $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$.

Examples

Example 4 Work out the length of side w .
Give your answer correct to 3 significant figures.



$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$w^2 = 8^2 + 7^2 - 2 \times 8 \times 7 \times \cos 45^\circ$$

$$w^2 = 33.804\ 040\ 51\dots$$

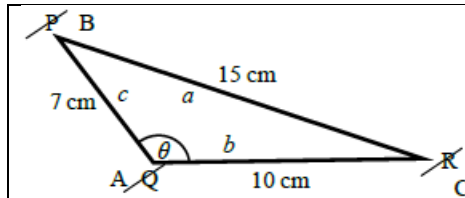
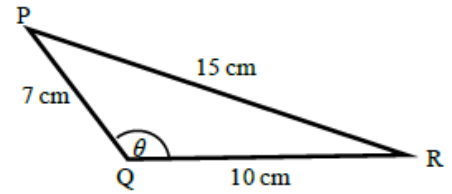
$$w = \sqrt{33.804\ 040\ 51}$$

$$w = 5.81 \text{ cm}$$

- 1 Always start by labelling the angles and sides.
- 2 Write the cosine rule to find the side.
- 3 Substitute the values a , b and A into the formula.
- 4 Use a calculator to find w^2 and then w .
- 5 Round your final answer to 3 significant figures and write the units in your answer.



Example 5 Work out the size of angle ϑ .
Give your answer correct to 1 decimal place.



$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos \theta = \frac{10^2 + 7^2 - 15^2}{2 \times 10 \times 7}$$

$$\cos \theta = \frac{-76}{140}$$

$$\vartheta = 122.878\ 349\dots$$

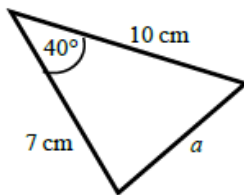
$$\vartheta = 122.9^\circ$$

- 1 Always start by labelling the angles and sides.
- 2 Write the cosine rule to find the angle.
- 3 Substitute the values a , b and c into the formula.
- 4 Use \cos^{-1} to find the angle.
- 5 Use your calculator to work out $\cos^{-1}(-76 \div 140)$.
- 6 Round your answer to 1 decimal place and write the units in your answer.

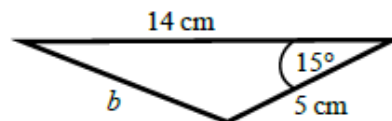
Practice

6 Work out the length of the unknown side in each triangle.
Give your answers correct to 3 significant figures.

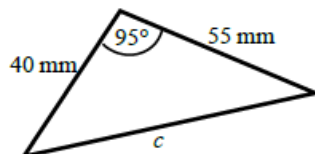
a



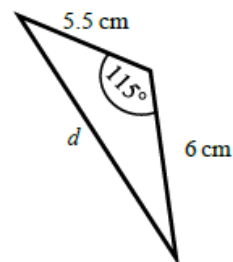
b



c

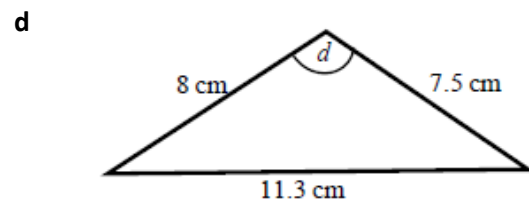
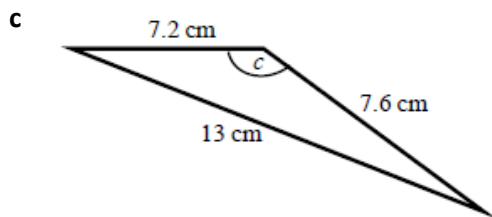
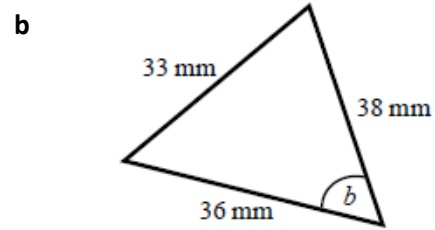
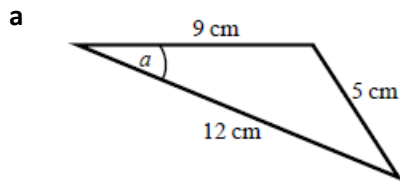


d

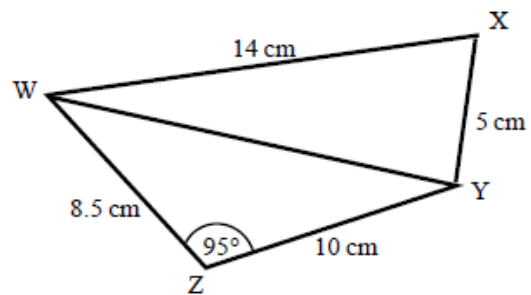




- 7 Calculate the angles labelled ϑ in each triangle. Give your answer correct to 1 decimal place.



- 8 a Work out the length of WY. Give your answer correct to 3 significant figures.
- b Work out the size of angle WXY. Give your answer correct to 1 decimal place.





The sine rule

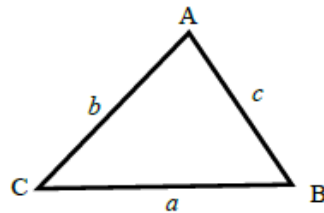
A LEVEL LINKS

Scheme of work: 4a. Trigonometric ratios and graphs

Textbook: Pure Year 1, 9.2 The sine rule

Key points

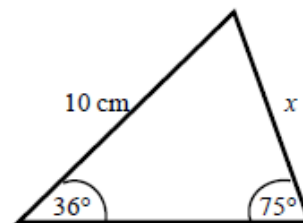
- a is the side opposite angle A .
- b is the side opposite angle B .
- c is the side opposite angle C .



- You can use the sine rule to find the length of a side when its opposite angle and another opposite side and angle are given.
- To calculate an unknown side use the formula $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.
- Alternatively, you can use the sine rule to find an unknown angle if the opposite side and another opposite side and angle are given.
- To calculate an unknown angle use the formula $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Examples

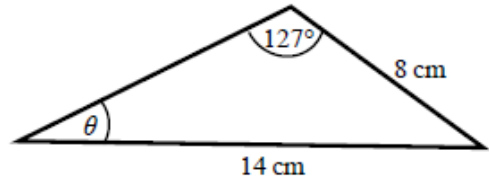
Example 6 Work out the length of side x .
Give your answer correct to 3 significant figures.

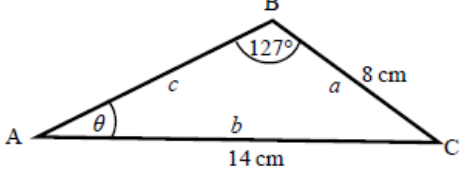


$\frac{a}{\sin A} = \frac{b}{\sin B}$ $\frac{x}{\sin 36^\circ} = \frac{10}{\sin 75^\circ}$ $x = \frac{10 \times \sin 36^\circ}{\sin 75^\circ}$ $x = 6.09 \text{ cm}$	<ol style="list-style-type: none">1 Always start by labelling the angles and sides.2 Write the sine rule to find the side.3 Substitute the values a, b, A and B into the formula.4 Rearrange to make x the subject.5 Round your answer to 3 significant figures and write the units in your answer.
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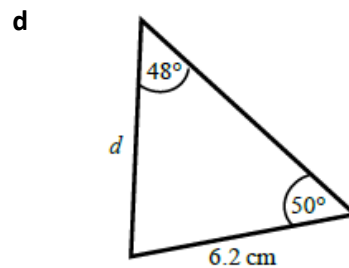
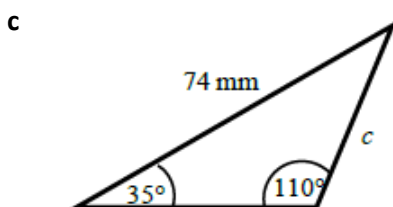
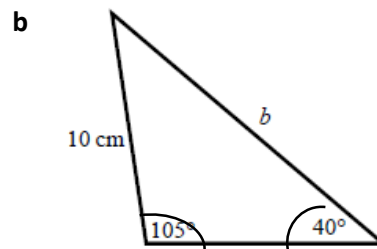
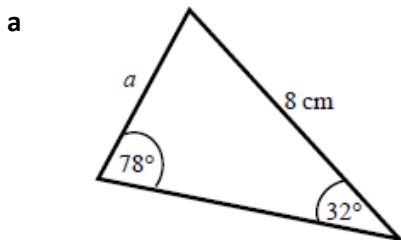
Example 7 Work out the size of angle ϑ .
Give your answer correct to 1 decimal place.



 $\frac{\sin A}{a} = \frac{\sin B}{b}$ $\frac{\sin \theta}{8} = \frac{\sin 127^\circ}{14}$ $\sin \theta = \frac{8 \times \sin 127^\circ}{14}$ $\theta = 27.2^\circ$	<ol style="list-style-type: none">1 Always start by labelling the angles and sides.2 Write the sine rule to find the angle.3 Substitute the values a, b, A and B into the formula.4 Rearrange to make $\sin \vartheta$ the subject.5 Use \sin^{-1} to find the angle. Round your answer to 1 decimal place and write the units in your answer.
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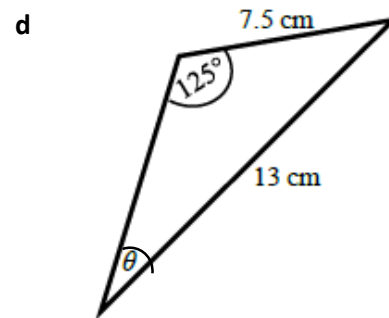
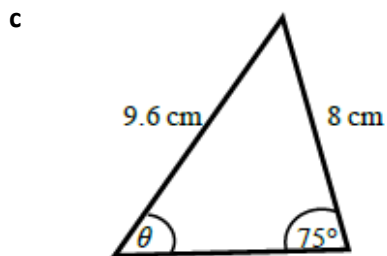
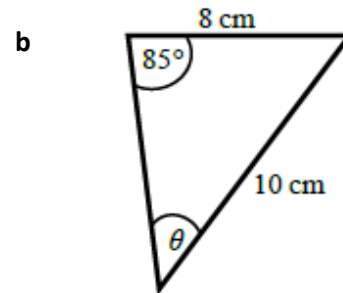
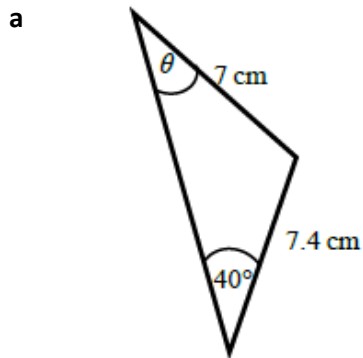
Practice

9 Find the length of the unknown side in each triangle.
Give your answers correct to 3 significant figures.

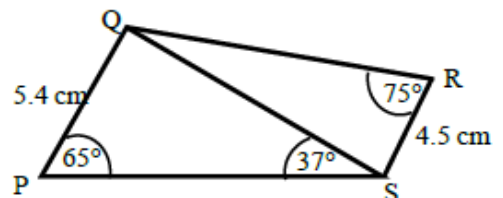




- 10 Calculate the angles labelled θ in each triangle.
Give your answer correct to 1 decimal place.



- 11 a Work out the length of QS.
Give your answer correct to 3 significant figures.
- b Work out the size of angle RQS.
Give your answer correct to 1 decimal place.





Areas of triangles

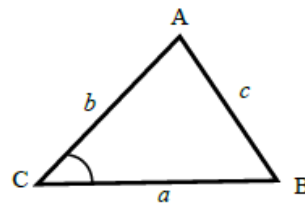
A LEVEL LINKS

Scheme of work: 4a. Trigonometric ratios and graphs

Textbook: Pure Year 1, 9.3 Areas of triangles

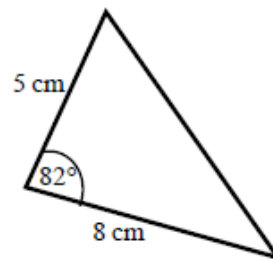
Key points

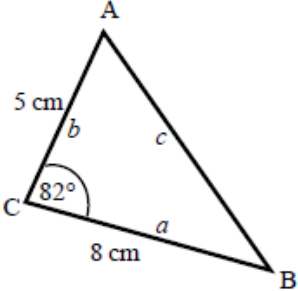
- a is the side opposite angle A.
 b is the side opposite angle B.
 c is the side opposite angle C.
- The area of the triangle is $\frac{1}{2}ab \sin C$.



Examples

Example 8 Find the area of the triangle.



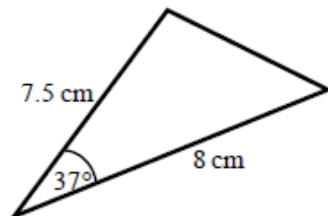
 <p>Area = $\frac{1}{2}ab \sin C$</p> <p>Area = $\frac{1}{2} \times 8 \times 5 \times \sin 82^\circ$</p> <p>Area = 19.805 361...</p> <p>Area = 19.8 cm²</p>	<ol style="list-style-type: none">1 Always start by labelling the sides and angles of the triangle.2 State the formula for the area of a triangle.3 Substitute the values of a, b and C into the formula for the area of a triangle.4 Use a calculator to find the area.5 Round your answer to 3 significant figures and write the units in your answer.
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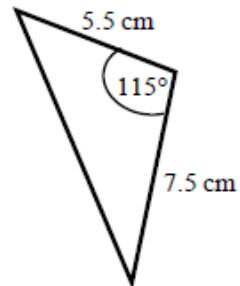
Practice

- 12 Work out the area of each triangle.
Give your answers correct to 3 significant figures.

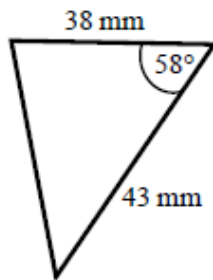
a



b



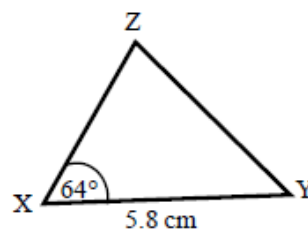
c



- 13 The area of triangle XYZ is 13.3 cm^2 .
Work out the length of XZ.

Hint:

Rearrange the formula to make a side the subject.

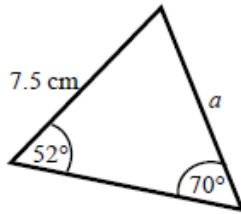




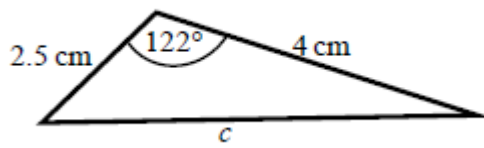
Extend

- 14 Find the size of each lettered angle or side.
Give your answers correct to 3 significant figures.

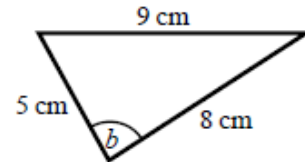
a



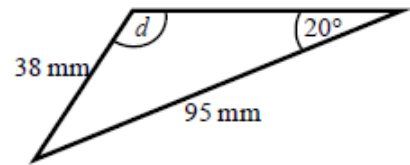
c



b



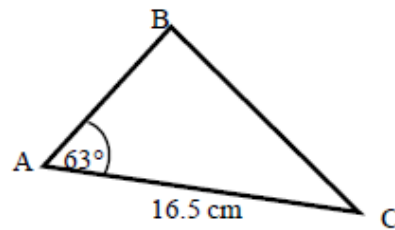
d



Hint:

For each one, decide whether to use the cosine or sine rule.

- 15 The area of triangle ABC is 86.7 cm^2 .
Work out the length of BC.
Give your answer correct to 3 significant figures.





Answers

1 a 6.49 cm
d 74.3 mm

b 6.93 cm
e 7.39 cm

c 2.80 cm
f 6.07 cm

2 a 36.9°

b 57.1°

c 47.0°

d 38.7°

3 5.71 cm

4 20.4°

5 a 45°

b 1 cm

c 30°

d $\sqrt{3}$ cm

6 a 6.46 cm

b 9.26 cm

c 70.8 mm

d 9.70 cm

7 a 22.2°

b 52.9°

c 122.9°

d 93.6°

8 a 13.7 cm

b 76.0°

9 a 4.33 cm

b 15.0 cm

c 45.2 mm

d 6.39 cm

10 a 42.8°

b 52.8°

c 53.6°

d 28.2°

11 a 8.13 cm

b 32.3°

12 a 18.1 cm²

b 18.7 cm²

c 693 mm²

13 5.10 cm

14 a 6.29 cm

b 84.3°

c 5.73 cm

d 58.8°

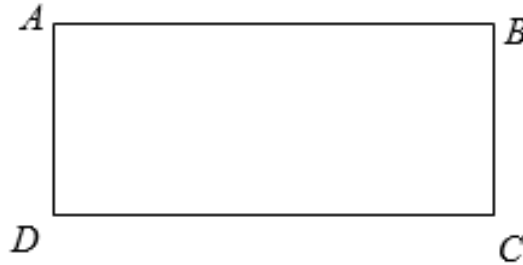
15 15.3 cm



PROBLEM-SOLVING TASKS

Question 1

The diagram shows a rectangle $ABCD$.



$$AB = 3^{2x-1} \text{ cm}$$

$$BC = 9 \text{ cm}$$

The area of the rectangle is 243 cm^2 .

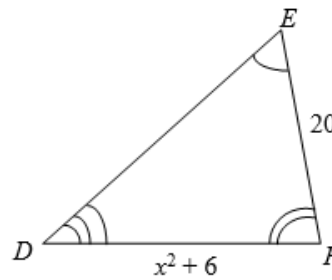
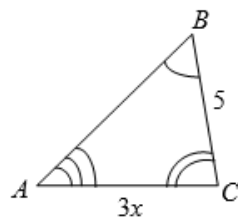
Work out the value of x .

You must show your working.

Question 2

Triangle ABC is mathematically similar to triangle DEF .

All lengths are in centimetres.



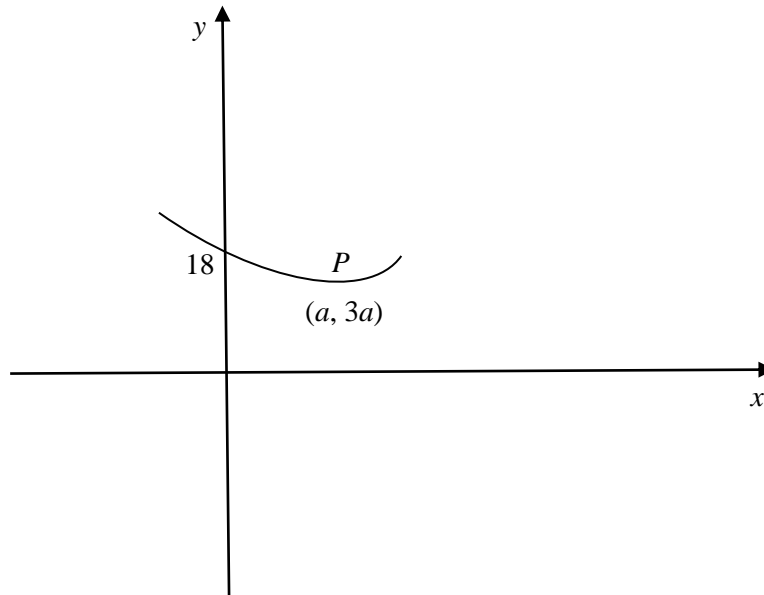
BC is the shortest side of triangle ABC .

Work out the exact value of x in the form $a + \sqrt{b}$.



Question 3

The diagram shows a sketch of $y = x^2 + bx + c$



The curve intersects the y -axis at $(0,18)$.

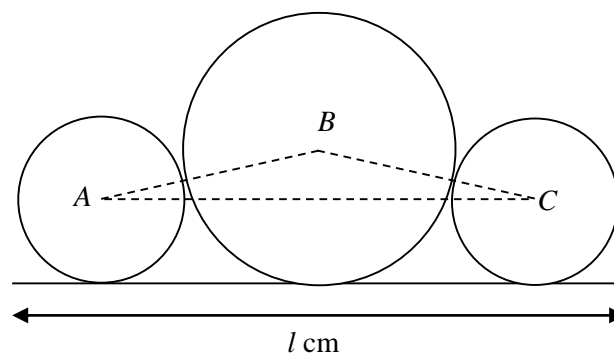
The point P with coordinates $(a, 3a)$ is the turning point of the curve.

Work out the value of a .

You must show your working.

Question 4

The diagram shows three circles.



The circle with centre A has a radius 4 cm.

The circle with centre B has a radius 8 cm.

The circle with centre C has a radius 4 cm.

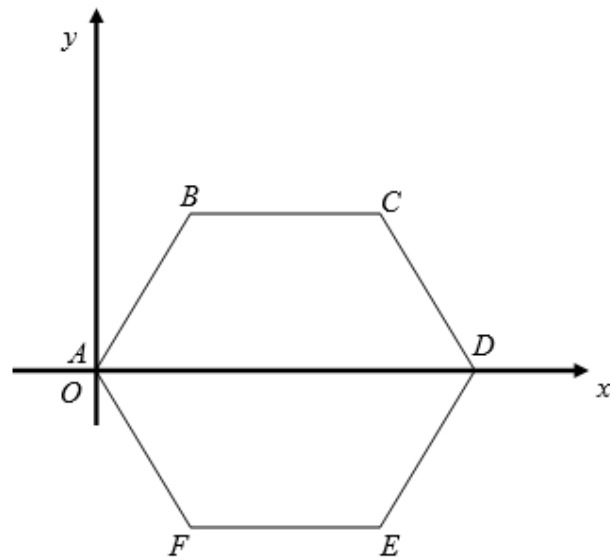
Work out the length, l , in cm.

Give your answer in the form $a + b\sqrt{2}$ where a and b are integers.



Question 5

$ABCDEF$ is a regular hexagon.

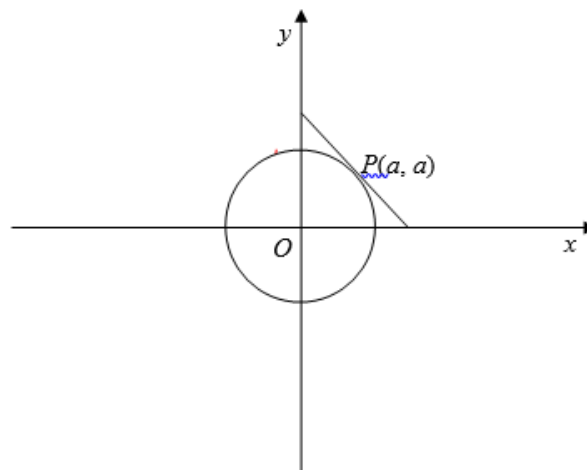


A is the point $(0, 0)$

B is the point $(2, 2\sqrt{3})$

Show that the area of the hexagon is $a\sqrt{3}$ where a is an integer to be found.

Question 6



The diagram shows a circle, centre O .

It also shows a tangent to the circle at the point $P(a, a)$.

The radius of the circle is $6\sqrt{2}$ cm.

Work out the coordinates of P .

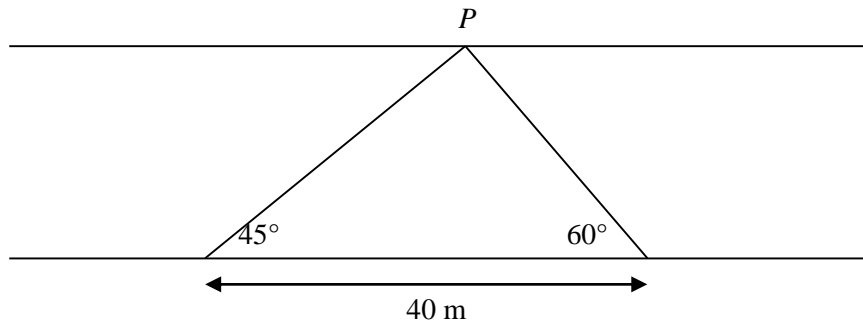
You must show your working.



Question 7

David wants to measure the width of a road

He chooses two points, A and B , 40 m apart.



The angles made with a lamppost, P , on the opposite side are measured.

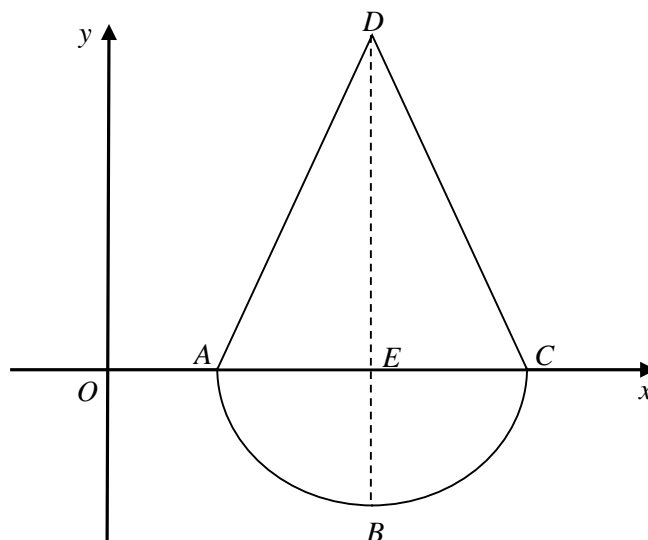
Angle $BAP = 45^\circ$

Angle $ABP = 60^\circ$

Work out the width, in metres, of the road. Give your answer correct to 3 significant figures.

Question 7

Here is a sketch of a cross section of a toy.



The curve has equation $y = x^2 - 14x + 40$ where x and y are measured in centimetres.

BED is a straight line.

The ratio of the length BE to the length ED is 3:5

Work out the area of triangle ADC .